SECTION 10.4: THE DIVERGENCE AND INTEGRAL TESTS

As mentioned in the previous section, with rare exception, we will be unable to determine the limit of convergent series. Instead, we will settle for tests which will help us determine **if** a series converges or not. With that said, we will often suppress the starting index of a series **since the inclusion of, or deletion of, a finite number of terms will not affect the convergence or divergence of a series**.[1]

More specifically, if $\sum_{k=1}^{\infty} a_k$ converges, then so would $\sum_{k=100}^{\infty} a_k$. Likewise if $\sum_{k=1}^{\infty} a_k$ diverges, so would $\sum_{k=100}^{\infty} a_k$.

Hence, you will often seen generic series represented as $\sum_{k}^{\infty} a_k$ with no starting index.

Suppose $\sum_{k}^{\infty} a_k$ converges. More specifically, suppose $\sum_{k}^{\infty} a_k = L$.

By definition, this means the sequence of partial sums converge to L: $\lim_{n\to\infty} S_n = L$.

Since S_n is the sum of the first n terms, and S_{n-1} represents the sum of the first $n-1$ terms, $S_n = S_{n-1} + a_n$.

Since $\lim_{n\to\infty} S_n = L$, it follows that $\lim_{n\to\infty} S_{n-1} = L$ (do you see why?)

Hence, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0$

We have shown that **if** $\sum_{k}^{\infty} a_k$ **converges**, **then** $\lim_{n\to\infty} a_n = 0$. It follows that **if** $\lim_{n\to\infty} a_n \neq 0$, **then** $\sum_{k}^{\infty} a_k$ **diverges**.

THE DIVERGENCE TEST: If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k}^{\infty} a_k$ diverges.

NOTE: As we'll see, if $\lim_{k\to\infty} a_k = 0$, the series $\sum_{k}^{\infty} a_k$ may converge or may diverge! Therefore remember:

If $\lim_{k\to\infty} a_k = 0$, the Divergence Test is inconclusive and we know nothing!

EXAMPLE 1: Apply the Divergence Test to the series below. What, if anything, can you conclude?

1. $\sum_{k=2}^{\infty} \frac{2k}{k-1}$. Ans: $\lim_{k\to\infty} a_k = 2$; series diverges.

2. $\sum_{k=0}^{\infty} \frac{1}{k!}$. Ans: $\lim_{k\to\infty} a_k = 0$; test inconclusive.

3. $\sum_{k=1}^{\infty} \frac{1}{k}$. Ans: $\lim_{k\to\infty} a_k = 0$; test inconclusive.

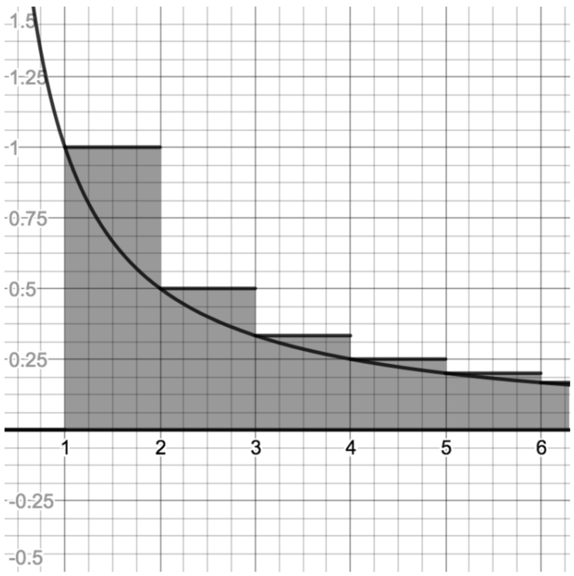
4. $\sum_{k=1}^{\infty} \cos(3k)$. Ans: $\lim_{k\to\infty} a_k$ does not exist; series diverges.

[1] Take a minute and think as to why this is the case.

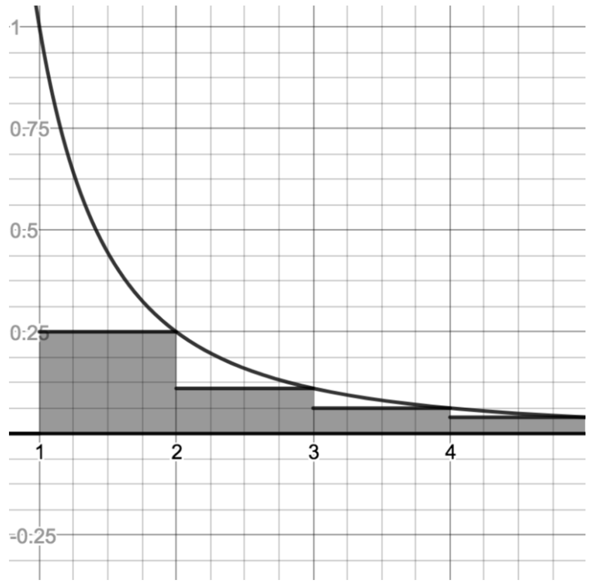
Let's take a moment to study the series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + ...$ This is the famous **harmonic series**.

As we saw in the previous example, the Divergence Test is inconclusive for this series and since the series isn't geometric or telescoping, we'll investigate this series in a different way.

Below on the left is the graph of $y = \frac{1}{x}$ along with a depiction of the left endpoint sum over the interval $[1, \infty)$:



left endpoint sum for $y = \frac{1}{x}$



right endpoint sum for $y = \frac{1}{x^2}$

The sum of the areas of these rectangles is greater than the area beneath $y = \frac{1}{x}$ for $x \geq 1$ which means:

$$1 + \frac{1}{2} + \frac{1}{3} + ... = \sum_{k=1}^{\infty} \frac{1}{k} \geq \int_1^{\infty} \frac{1}{x}\, dx$$

However, we know that $\int_1^{\infty} \frac{1}{x}\, dx = \infty$ which means $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Next, consider the series: $\sum_{k=1}^{\infty} \frac{1}{k^2}$. We also have $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k^2} = 0$, so the Divergence Test is inconclusive.

Above on the right, we look at the right endpoint sum of $y = \frac{1}{x^2}$ on $[1, \infty)$. Here we get:

$$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... \leq \int_1^{\infty} \frac{1}{x^2}\, dx = 1 \implies \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... \leq 1 + \int_1^{\infty} \frac{1}{x^2}\, dx = 2$$

Since all the terms of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ are positive, the sequence of partial sums is increasing.

The inequality above shows the partial sums are bounded above (by 2).

Hence, the sequence of partial sums (and therefore the series) converges by the completeness of the real numbers.

The arguments we used to prove $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges can be generalized to give us:

THE INTEGRAL TEST: If f is continuous, positive, and decreasing on some interval $[N,\infty)$, then

$$\int_N^{\infty} f(x)\,dx \quad \text{and} \quad \sum_{k=N}^{\infty} f(k)$$

either **both** converge or **both** diverge.

NOTE: The sum of the series and the value of the integral need not be equal. The integral test just says either both the sum and the integral will be finite numbers (in the convergent case) or become unbounded (∞) in the divergent case. Specifically, we showed that:

$$\sum_{k=1}^{\infty}\frac{1}{k^2} \leq 1 + \int_1^{\infty}\frac{1}{x^2}\,dx = 2,$$

but we can't conclude that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} = 2$.

EXAMPLE 2: Verify the following series satisfy the requirements of the integral test and use the integral test to determine if the series converges or diverges.

1. **(VIDEO)**: $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k+3}}$ Ans: diverges

2. **(VIDEO)**: $\displaystyle\sum_{k=1}^{\infty}\frac{k}{e^k}$ Ans: converges

3. $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k\,[\ln(k)]^2}$ Ans: converges

An immediate consequence of the Integral Test along with our work with p-series is the following:

p-**SERIES:** The series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^p}$ connverges if $p>1$ and diverges otherwise.

EXAMPLE 3: Determine if the series below converge or diverge. Explain your reasoning.

1. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k}}$ p-series, $p=\dfrac{3}{2}>1$; converges.

2. $\displaystyle\sum_{k=10}^{\infty}\frac{\sqrt[4]{k}}{k}$ finitely many terms different than a p-series with $p=\dfrac{3}{4}\leq 1$; diverges

3. $\displaystyle\sum_{k=1}^{\infty}\frac{3}{k}$ nonzero multiple of the harmonic series; diverges

REMAINDERS

As we mentioned, if a series converges by the integral test, the value of the sum and the value of the integral will likely be different. That being said, we can use the same right and left endpoint sums to help us **estimate** the sum of the series using improper integrals.

DEFINITION: Given a series $\sum_{k=1}^{\infty} a_k$ whose partial sums $\{S_n\}$ converge to S, the nth remainder, R_n is:

$$R_n = S - S_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = a_{n+1} + a_{n+2} + a_{n+3} + \ldots$$

That is, R_n is the error incurred by approximating the value of the series by the nth partial sum.

Using the usual geometric arguments, we get the following result:

INTEGRAL TEST REMAINDER THEOREM:

If $\sum_{k=1}^{\infty} a_k$ converges by the integral test by comparison with $\int_1^{\infty} f(x)\,dx$ then $R_n \leq \int_n^{\infty} f(x)\,dx$.

Moreover, $S_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_n^{\infty} f(x)\,dx$.

EXAMPLE 4: Consider the series: $\sum_{k=1}^{\infty} \frac{1}{k^3}$.

1. Show this series converges. Ans: p-series, $p = 3 > 1$; converges.

2. Approximate the value of the series by computing S_4. Ans: $1.177662\overline{037}$

3. Approximate the error incurred by approximating the sum by S_4. Ans: $R_4 \leq 0.03125$

4. Bound the value of the series. $1.197662\overline{037} \leq \sum_{k=1}^{\infty} \frac{1}{k^3} \leq 1.208912\overline{037}$

5. Find the smallest n so that S_n approximates the sum of the series to within 0.00001.

   Ans: $n = 224$; $S_{224} = 1.20204 \pm 0.00001$

For us, the main use of the integral test is that it gives us p-series and a way to estimate remainders of p-series. The integral test itself has its niche, of course, but it will often be a test of ‘last resort’ once we've seen all of the other tools this chapter has to offer.

HOMEWORK: Section 10.4: 9 - 39 odd, 47 - 65 odd, 67*